# 113 Class Problems: Finite Abelian Groups

All of these questions use the structure theorem

1. (a) Describe (up to isomorphism) all Abelian groups of size 15
   (b) Describe (up to isomorphism) all Abelian groups of size 125.

   **Solutions:**

a) $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$

b) $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$

$\mathbb{Z}/5^2\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$

$\mathbb{Z}/5^3\mathbb{Z}$

2. Describe (up to isomorphism) all Abelian groups of size 72, which contain an element of order 36.

   **Solutions:**

$\mathbb{Z}/2^2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3^2\mathbb{Z}$

$\mathbb{Z}/2^3\mathbb{Z} \times \mathbb{Z}/3^2\mathbb{Z}$

3. Let $G$ be an Abelian group of size $n$. Assume $n$ has distinct prime factors $\{p_1, \cdots, p_k\}$. Prove the following:

$$G_{p_i} \text{ cyclic } \forall i \quad \Rightarrow \quad G \text{ cyclic.}$$

**Solutions:** $|G| = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$

$G_{p_i}$ cyclic $\Rightarrow$ $G_{p_i} \cong \mathbb{Z}/p_i^{\alpha_i}\mathbb{Z}$

$\Rightarrow$ $G \cong \mathbb{Z}/p_1^{\alpha_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_k^{\alpha_k}$

$\text{ord}\left([1]_{p_1^{\alpha_1}}, \ldots, [1]_{p_k^{\alpha_k}}\right) = \text{LCM}(p_1^{\alpha_1}, \ldots, p_k^{\alpha_k}) = |G|$

$\Rightarrow$ $G$ cyclic

4. Let $G$ be Abelian with $|G| = n$. Prove that if $m|n$ then there exists a subgroup $H \subset G$ of such that $|H| = m$.

The result holds for any cyclic group.

A finite Abelian group is the direct product of cyclic groups. $G = G_1 \times G_2 \cdots \times G_r$

Select subgroups $H_1 \subset G_1, \ldots, H_r \subset G_r$ such that

$|H_1| \cdots |H_r| = m$

$H_1 \times H_2 \times \cdots \times H_r \subset G$ is a subgroup and

$|H_1 \times \cdots \times H_r| = m.$